Jacobi fields along harmonic maps[*]

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Abstract

We show that Jacobi fields along harmonic maps between suitable spaces preserve conformality, holomorphicity, real isotropy and complex isotropy to first order; this last being one of the key tools in the proof by Lemaire and the author of integrability of Jacobi fields along harmonic maps from the 2-sphere to the complex projective plane.

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1 Introduction

A *Jacobi field* (for the energy) along a harmonic map φ is a vector field along φ which is in the kernel of the second variation of the energy. Equivalently, it is tangent to a variation of φ for which the tension field remains zero 'to first order'. We shall show that Jacobi fields preserve several properties of a harmonic map to first order, namely holomorphicity for maps between compact Kähler manifolds, conformality of maps from the 2-sphere, real isotropy of a harmonic map from the 2-sphere to a space form, and complex isotropy of a harmonic map from the 2-sphere to a complex space form. The main idea of the proofs is to define differentials depending on the map and the Jacobi field which are shown to be holomorphic and so vanish; this generalizing the proof of isotropy of a harmonic map from the 2-sphere as given, for example, in [22].

Note that any harmonic map from the 2-sphere is weakly conformal and so is the same thing as a *minimal branched immersion* in the sense of [12].

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Lastly, we mention that the Jacobi equation is preserved by *harmonic morphisms*, see [19] for this result, [11] for a bibliography and [1] for an account of the theory of harmonic morphisms.

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2   Jacobi fields

Throughout this paper $M = (M, g)$ and $N = (N, h)$ will denote smooth (i.e. C^∞) compact Riemannian manifolds without boundary and $\varphi : M \to N$ will denote a smooth map. For any vector bundle $E \to M$, $\Gamma(E)$ will denote the space of smooth sections of E. Particularly important is the *pull-back bundle* $\varphi^{-1}TN \to M$; its smooth sections are called *vector fields along φ*.

We recall some basic definitions, see [8] for more details. The *energy* of φ is defined by the integral

$$E(\varphi) = \tfrac{1}{2} \int_M |\mathrm{d}\varphi|^2 \, \omega_g$$

where $|\mathrm{d}\varphi|$ denotes the Hilbert–Schmidt norm of the differential of φ and ω_g denotes the volume form of the metric g. By a *(smooth) (1-parameter) variation* $\{\varphi_t\}$ of φ we mean a smooth map $M \times I \to N$, $(x, t) \mapsto \varphi_t(x)$, where I is an open interval of the real line containing 0, such that $\varphi_0 = \varphi$.

Given a smooth variation $\{\varphi_t\}$, we set

$$v = \left. \frac{\partial \varphi_t}{\partial t} \right|_0$$

where $\left. \frac{\partial}{\partial t} \right|_0$ denotes (partial) derivative with respect to t at $t = 0$; this defines a vector field along φ called the *variation vector field of $\{\varphi_t\}$*. Then, for any smooth map $\varphi : M \to N$ and any smooth variation $\{\varphi_t\}$ of it we have

$$\left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_0 E(\varphi_t) = - \int_M \langle \tau(\varphi), v \rangle \, \omega_g \tag{1}$$

where $\tau(\varphi)$ denotes the vector field along φ called the *tension field* of φ given by $\tau(\varphi) = \mathrm{Trace}\, \nabla \mathrm{d}\varphi$.

Here and in the sequel, $\langle \; , \; \rangle$ and ∇ denote the inner product and connection induced on the relevant bundle by the metrics and Levi-Civita connections on M and N, see, for example, [8] for this formalism.

The formula (1) shows that the number $\frac{\mathrm{d}}{\mathrm{d}t}\big|_0 E(\varphi_t)$ depends only on the variation vector field v of $\{\varphi_t\}$. Given any $v \in \Gamma(\varphi^{-1}TN)$, there are infinitely many smooth variations $\{\varphi_t\}$ *tangent to* (i.e. *with variation vector field*) v. Writing the left-hand side of (1) as $\nabla_v E$ gives the *first variation formula for the energy*:

$$\nabla_v E = -\int_M \langle \tau(\varphi), v \rangle \, \omega_g \,. \tag{2}$$

A smooth map φ is called *harmonic* if the first variation (2) is zero for all $v \in \Gamma(\varphi^{-1}TN)$; equivalently, from (2), φ satisfies the equation

$$\tau(\varphi) = 0 \,. \tag{3}$$

In local coordinates this is an elliptic semilinear system, not linear except when (N, h) is flat.

Now suppose that $\varphi : M \to N$ is harmonic. Then we can define its *second variation* as follows. By a *(smooth) 2-parameter variation* $\{\varphi_{t,s}\}$ of φ we mean a smooth map $M \times I^2 \to N$, $(x, t, s) \mapsto \varphi_{t,s}(x)$, where I^2 is an open connected subset of $\mathbb{R}^2$ containing $(0,0)$, such that $\varphi_{0,0} = \varphi$. The *Hessian of* φ is defined on a pair v, w of vector fields along φ by

$$H_\varphi(v, w) = \left. \frac{\partial^2 E(\varphi_{t,s})}{\partial t \partial s} \right|_{(0,0)} \tag{4}$$

where $\{\varphi_{t,s}\}$ is a smooth 2-parameter variation of φ with

$$\left. \frac{\partial \varphi_{t,s}}{\partial t} \right|_{(0,0)} = v \quad \text{and} \quad \left. \frac{\partial \varphi_{t,s}}{\partial s} \right|_{(0,0)} = w \,.$$

Then the left-hand side of (4) depends only on v and w and is given by

$$H_\varphi(v, w) = \int_M \langle J_\varphi(v), w \rangle \, \omega_g \tag{5}$$

where

$$J_\varphi(v) = \Delta v - \operatorname{Trace} R^N(\mathrm{d}\varphi, v)\mathrm{d}\varphi \,;$$

here Δ denotes the Laplacian on $\varphi^{-1}TN$ and R^N the curvature operator of N (conventions as in [8]). The mapping $J_\varphi : \Gamma(\varphi^{-1}TN) \to \Gamma(\varphi^{-1}TN)$ is called the *Jacobi operator* (for the energy); it is a self-adjoint elliptic *linear* operator. A vector field v along φ is called a *Jacobi field (along φ)* if it is in the kernel of the Jacobi operator, i.e., it satisfies the *Jacobi equation*

$$J_\varphi(v) \equiv \Delta v - \operatorname{Trace} R^N(\mathrm{d}\varphi, v)\mathrm{d}\varphi = 0 \,. \tag{6}$$

By standard elliptic theory (see, for example, [18]) the set of Jacobi fields along a harmonic map is a finite dimensional vector subspace of $\Gamma(\varphi^{-1}TN)$.

We shall make use of the following interpretation of the Jacobi operator as the *linearization* of the tension field.

Proposition 2.1 *Let* $\varphi : M \to N$ *be harmonic and let* $v \in \Gamma(\varphi^{-1}TN)$. *Let* $\{\varphi_t\}$ *be a smooth variation of* φ *tangent to* v. *Then*

$$J_\varphi(v) = -\left.\frac{\partial}{\partial t}\right|_0 \tau(\varphi_t) \,.$$

In particular, v *is a Jacobi field along* φ *if and only if the tension field of* φ_t *is zero to first order, i.e.,*

$$\left.\frac{\partial}{\partial t}\right|_0 \tau(\varphi_t) = \tau(\varphi) = 0 \,. \tag{7}$$

Proof From (4) we have

$$\begin{aligned}
H_\varphi(v, w) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \left(\nabla_w E(\varphi_{t,0})\right) \\
&= -\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_0 \int_M \langle \tau(\varphi_{t,0}), w \rangle \, \omega_g \\
&= \int_M \left\langle -\left.\frac{\partial}{\partial t}\right|_0 \tau(\varphi_{t,0}), w \right\rangle \omega_g \,.
\end{aligned}$$

Comparing this with (5) gives the statement. $\square$

We shall write the condition (7) succinctly as

$$\tau(\varphi_t) = o(t) \,; \tag{8}$$

and we shall call a smooth variation $\{\varphi_t\}$ *harmonic to first order* if it satisfies this condition. Then the Proposition tells us that *a smooth variation* $\{\varphi_t\}$ *of a harmonic map* φ *is harmonic to first order if and only if it is tangent to a Jacobi field along* φ.

In particular, if $\{\varphi_t\}$ is a smooth variation of φ through harmonic maps, its variation vector field $v = \left.\dfrac{\partial \varphi_t}{\partial t}\right|_0$ is a Jacobi field. Conversely, a Jacobi field along a harmonic map is called *integrable* if it is tangent to a variation through harmonic maps. Not all Jacobi fields are integrable as well-known examples along geodesics show; see also [20] for two-dimensional examples.

Integrability has applications to the study of spaces of harmonic maps and to the behaviour of a minimizing harmonic map near a singularity, see [17] and the references therein. It is thus an important question to determine for what pairs of Riemannian manifolds all Jacobi fields along harmonic maps are integrable. The known examples are very few in number; recently L. Lemaire and the author have added the case $M = S^2$, $N = \mathbb{C}P^2$ [17]. One main idea in the proof is that many properties of a harmonic map are 'preserved to first order' under a variation tangent to a Jacobi field. We shall explain this in the next two sections.

3 Preservation of conformality and isotropy to first order

We discuss two properties of a harmonic map from a 2-sphere to a Riemannian manifold which are preserved to first order under smooth variations tangent to a Jacobi field; the first, *conformality*, involves only the first order partial derivatives of the map, the second, *real isotropy* is a stronger condition involving higher order partial derivatives.

3.1 Preservation of conformality

Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. Then φ is called *weakly conformal* if, for each $x \in M$ there is a number $\lambda(x) \in [0, \infty)$ such that $|\mathrm{d}\varphi_x(X)| = \lambda(x)|X|$ for all $X \in T_x M$.

To formulate this in a way we can use, it is convenient to extend the inner product on TN to a complex-bilinear inner product $\langle \ , \ \rangle^{\mathbf{C}}$ on the complexified tangent space $T^{\mathbf{c}} N = TN \otimes_{\mathbb{R}} \mathbb{C}$ (and so also on $\varphi^{-1} T^{\mathbf{c}} N = \varphi^{-1} TN \otimes_{\mathbb{R}} \mathbb{C}$). We also extend the differential of φ to a complex-linear map $\mathrm{d}^{\mathbf{c}} \varphi : T^{\mathbf{c}} M \to T^{\mathbf{c}} N$ between complexified tangent spaces.

Suppose now that M^2 is a Riemann surface, i.e. a 1-dimensional complex manifold. For any complex coordinate z on M^2 write

$$\frac{\partial^{\mathbf{c}} \varphi}{\partial z} = (\mathrm{d}^{\mathbf{c}} \varphi)\Big(\frac{\partial}{\partial z}\Big), \quad \frac{\partial^{\mathbf{c}} \varphi}{\partial \bar{z}} = (\mathrm{d}^{\mathbf{c}} \varphi)\Big(\frac{\partial}{\partial \bar{z}}\Big).$$

Then φ is weakly conformal if and only if

$$\eta^{\mathbf{R}}(\varphi) \equiv \Big\langle \frac{\partial^{\mathbf{c}} \varphi}{\partial z}, \frac{\partial^{\mathbf{c}} \varphi}{\partial z} \Big\rangle^{\mathbf{C}} = 0 \,,$$

this condition being independent of the choice of complex coordinate; it is equivalent to the vanishing of the $(2, 0)$-part of the pull-back $\varphi^{-1} h$ of the metric on N.

Extend the connection ∇ on $\varphi^{-1}TN$ by complex-bilinearity to $\varphi^{-1}T^{\mathbf{c}}N$ and write

$$\frac{\nabla}{\partial z} = \nabla_{\frac{\partial}{\partial z}}\,, \quad \frac{\nabla}{\partial \bar{z}} = \nabla_{\frac{\partial}{\partial \bar{z}}}\,.$$

Then, a smooth map $\varphi : M^2 \to N$ is harmonic if and only if

$$\frac{\nabla}{\partial \bar{z}}\frac{\partial^{\mathbf{c}}\varphi}{\partial z} = 0\,, \quad \text{equivalently,} \quad \frac{\nabla}{\partial z}\frac{\partial^c\varphi}{\partial \bar{z}} = 0\,, \tag{9}$$

these expressions giving a non-zero multiple of the tension field with respect to any Hermitian metric on M^2; we thus recover the well-known result that harmonicity is independent of the choice of such a metric.

Now suppose $\varphi : M^2 \to N$ is harmonic. Then, by the first expression in (9), a smooth variation $\{\varphi_t\}$ of φ is harmonic to first order if and only if

$$\frac{\nabla}{\partial t}\Big|_0 \frac{\nabla}{\partial \bar{z}}\frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} = 0\,; \tag{10}$$

swapping derivatives using the curvature of N shows that this is equivalent to

$$\frac{\nabla}{\partial \bar{z}}\frac{\nabla}{\partial z}\frac{\partial\varphi_t}{\partial t}\Big|_0 + R\Big(\frac{\partial^c\varphi}{\partial \bar{z}}, \frac{\partial\varphi_t}{\partial t}\Big|_0\Big)\frac{\partial^{\mathbf{c}}\varphi}{\partial z} = 0\,.$$

It follows that $v \in \Gamma(\varphi^{-1}TN)$ is a Jacobi field along φ if and only if

$$\frac{\nabla}{\partial \bar{z}}\frac{\nabla}{\partial z}v + R\Big(\frac{\partial^c\varphi}{\partial \bar{z}}, v\Big)\frac{\partial^{\mathbf{c}}\varphi}{\partial z} = 0 \tag{11}$$

or, equivalently (the conjugate of (11)),

$$\frac{\nabla}{\partial z}\frac{\nabla}{\partial \bar{z}}v + R\Big(\frac{\partial^{\mathbf{c}}\varphi}{\partial z}, v\Big)\frac{\partial^c\varphi}{\partial \bar{z}} = 0\,.$$

We remark that these equations could be obtained directly from (6). Indeed, their sum is that equation and their difference is a consequence of the first Bianchi identity for the curvature.

Now let $\varphi : M^2 \to N$ be a smooth map and let $\{\varphi_t\}$ be a smooth variation of it. For any complex coordinate, set

$$\eta^{\mathbf{R}}(\varphi_t) = \Big\langle \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z}, \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} \Big\rangle^{\mathbf{C}}\,.$$

Say that $\{\varphi_t\}$ is *conformal to first order* if $\eta^{\mathbf{R}}(\varphi_t) = o(t)$, i.e.

$$\frac{\partial}{\partial t}\Big|_0 \eta^{\mathbf{R}}(\varphi_t) = \eta^{\mathbf{R}}(\varphi) = 0\,.$$

Let $v = \left.\dfrac{\partial \varphi_t}{\partial t}\right|_0$ be the variation vector field of $\{\varphi_t\}$. Then, since ∇ is a Riemannian connection,

$$\left.\frac{\partial}{\partial t}\right|_0 \eta^{\mathbf{R}}(\varphi_t) = 2\left\langle \frac{\nabla v}{\partial z}, \frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}}. \tag{12}$$

We call a vector field v along a smooth map $\varphi : M^2 \to N$ *conformal* if $\left\langle \dfrac{\nabla v}{\partial z}, \dfrac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} = 0$; this condition is clearly independent of the choice of complex coordinate. Then from (12) we see that conformality to first order is equivalent to conformality of the variation vector field as follows:

Lemma 3.1 *Let $\varphi : M^2 \to N$ be a weakly conformal map from a Riemann surface to a Riemannian manifold and let $v \in \Gamma(\varphi^{-1}TN)$. Then the following are equivalent:*

  (i) *v is a conformal vector field along φ;*

  (ii) *there exists a smooth variation of φ tangent to v which is conformal to first order;*

  (iii) *all smooth variations of φ tangent to v are conformal to first order.*

$\hfill\square$

Any harmonic map from the 2-sphere S^2 to an arbitrary Riemannian manifold N is weakly conformal. This is proved by showing that, for any harmonic map $\varphi : M^2 \to N$ from a Riemann surface, $\eta^{\mathbf{R}}(\varphi)\,\mathrm{d}z^2$ defines a *holomorphic quadratic differential*, i.e. a holomorphic section of the holomorphic line bundle $\otimes^2 T'_* M^2$; if $M^2 = S^2$, this bundle has negative degree and so any holomorphic section vanishes, see, for example, [21]. We generalize this in the next result; for later comparison we give two proofs.

Proposition 3.2 *Any Jacobi field along a harmonic map $\varphi : S^2 \to N$ is conformal; equivalently, any smooth variation of φ tangent to a Jacobi field is conformal to first order.*

Proof 1 Let $\varphi : M^2 \to N$ be a harmonic map from a Riemann surface and let v be a Jacobi field along it. Using the Jacobi equation (11), in any complex coordinate we have

$$\begin{aligned}
\frac{\partial}{\partial \bar{z}}\left\langle \frac{\nabla v}{\partial z}, \frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} &= \left\langle \frac{\nabla}{\partial \bar{z}}\frac{\nabla v}{\partial z}, \frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} + \left\langle \frac{\nabla v}{\partial z}, \frac{\nabla}{\partial \bar{z}}\frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} \\
&= -\left\langle R\Big(\frac{\partial^{\mathbf{c}}\varphi}{\partial \bar{z}}, v\Big)\frac{\partial^{\mathbf{c}}\varphi}{\partial z}, \frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} + 0 \\
&= 0\,.
\end{aligned}$$

Hence $\left\langle \dfrac{\nabla v}{\partial z}, \dfrac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} \mathrm{d}z^2$ defines a holomorphic differential on M^2; if $M^2 = S^2$ this must vanish.

Proof 2 Let $\varphi : M^2 \to N$ be a harmonic map from a Riemann surface and let $\{\varphi_t\}$ be a smooth variation of it tangent to a Jacobi field. Then, by Proposition 2.1,

$$\frac{\partial}{\partial \bar{z}} \eta^{\mathbf{R}}(\varphi_t) = 2\left\langle \frac{\nabla}{\partial \bar{z}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z}, \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} \right\rangle^{\mathbf{C}} = o(t) \,.$$

It follows that

$$\left.\frac{\partial}{\partial \bar{z}} \frac{\partial}{\partial t}\right|_0 \eta^{\mathbf{R}}(\varphi_t) = \left.\frac{\partial}{\partial t}\right|_0 \frac{\partial}{\partial \bar{z}} \eta^{\mathbf{R}}(\varphi_t) = 0 \,.$$

Hence, as well as $\eta^{\mathbf{R}}(\varphi)$, $\left.\dfrac{\partial}{\partial t}\right|_0 \eta^{\mathbf{R}}(\varphi_t)$ is a holomorphic quadratic differential and so must vanish if $M^2 = S^2$. $\qquad\square$

3.2   Preservation of real isotropy

For any smooth map $\varphi : M^2 \to N$ from a Riemann surface to a Riemannian manifold, integers $r, s \geq 1$ and local complex coordinate z, set

$$\eta^{\mathbf{R}}_{r,s}(\varphi) = \left\langle \frac{\nabla^{r-1}}{\partial z^{r-1}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z}, \frac{\nabla^{s-1}}{\partial z^{s-1}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} .$$

Recall that φ is called *real isotropic* or *pseudoholomorphic* [5] if $\eta^{\mathbf{R}}_{r,s}(\varphi) = 0$ for all integers $r, s \geq 1$. Note that this condition is independent of the complex coordinate chosen and that any real isotropic map is weakly conformal.

Let $\varphi : S^2 \to N$ be a harmonic map from the 2-sphere to a *space form*, i.e. a Riemannian manifold of constant sectional curvature. Then [4, 5] φ is real isotropic. This is proved by showing, inductively that, for $k = 2, 3, \ldots$ and $r + s = k$, $\eta^{\mathbf{R}}_{r,s}(\varphi)\mathrm{d}z^k$ defines a *holomorphic k-differential,* i.e., a holomorphic section of $\otimes^k T'_* S^2$, and so must vanish. It follows that all harmonic maps can be constructed explicitly from holomorphic maps, see [5, 6, 2, 3, 15]. We now generalize this argument. Let $\{\varphi_t\}$ be a 1-parameter variation of φ. Say that $\{\varphi_t\}$ *is real isotropic to first order* if $\eta^{\mathbf{R}}_{r,s}(\varphi_t) = o(t)$ for all r, s, i.e.,

$$\left.\frac{\partial}{\partial t}\right|_0 \eta^{\mathbf{R}}_{r,s}(\varphi_t) = \eta^{\mathbf{R}}_{r,s}(\varphi) = 0 \qquad (r, s \geq 1) \,.$$

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Note that this condition is independent of the choice of complex coordinate. It can be formulated in terms of the variation vector field of $\{\varphi_t\}$ as follows. For any smooth map $\varphi : M^2 \to N$ from a Riemann surface to a Riemannian manifold, vector field v along φ, integers $r, s \geq 1$ and local complex coordinate z, set

$$j_{r,s}^{\mathbf{R}}(v) = \left\langle \frac{\nabla^r v}{\partial z^r}, \frac{\nabla^{s-1}}{\partial z^{s-1}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} + \left\langle \frac{\nabla^{r-1}}{\partial z^{r-1}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z}, \frac{\nabla^s v}{\partial z^s} \right\rangle^{\mathbf{C}}.$$

Then we have

Lemma 3.3 *Let* $\varphi : M^2 \to N$ *be a real isotropic map from a Riemann surface into a space form and let* v *be a Jacobi field along it. Then, for any smooth variation* $\{\varphi_t\}$ *of* φ *tangent to* v *and any complex coordinate* z *on* M^2 *we have*

$$j_{r,s}^{\mathbf{R}}(v) = \frac{\partial}{\partial t}\bigg|_0 \eta_{r,s}^{\mathbf{R}}(\varphi_t) \qquad (r, s \geq 1).$$

Proof We show by induction that, for $k = 1, 2, \ldots,$

$$\frac{\nabla}{\partial t}\bigg|_0 \frac{\nabla^{k-1}}{\partial z^{k-1}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} \in \theta_{k-1}(\varphi) + \frac{\nabla^k v}{\partial z^k} \qquad (13)$$

where

$$\theta_{k-1}(\varphi) = \operatorname{span}\left\{ \frac{\partial^{\mathbf{c}}\varphi}{\partial z}, \ldots, \frac{\nabla^{k-2}}{\partial z^{k-2}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\} \ (k \geq 2) \ \text{ and } \ \theta_0(\varphi) = \{\mathbf{0}\} \,;$$

note that this is independent of the choice of complex coordinate.

For $k = 1$, (13) holds since $\dfrac{\nabla}{\partial t}\bigg|_0 \dfrac{\partial^{\mathbf{c}}\varphi_t}{\partial z} = \dfrac{\nabla}{\partial z} \dfrac{\partial \varphi_t}{\partial t}\bigg|_0 = \dfrac{\nabla v}{\partial z}$.

Suppose that (13) holds for $k - 1$ for some $k \geq 2$. Then

$$\frac{\nabla}{\partial t}\bigg|_0 \frac{\nabla^{k-1}}{\partial z^{k-1}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} = \frac{\nabla}{\partial z} \frac{\nabla}{\partial t}\bigg|_0 \frac{\nabla^{k-2}}{\partial z^{k-2}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} + R\Big(\frac{\partial^{\mathbf{c}}\varphi}{\partial z}, v\Big) \frac{\nabla^{k-2}}{\partial z^{k-2}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z}. \quad (14)$$

Now, by the induction hypothesis the first term on the right-hand side of (14) lies in $\theta_{k-1}(\varphi) + \dfrac{\nabla^k v}{\partial z^k}$. Also, by the well-known formula for the curvature of a space form (see, for example, [13, Chapter V]), the curvature term is a multiple of

$$\left\langle v, \frac{\nabla^{k-2}}{\partial z^{k-2}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z} - \left\langle \frac{\partial^{\mathbf{c}}\varphi}{\partial z}, \frac{\nabla^{k-2}}{\partial z^{k-2}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} v \,; \qquad (15)$$

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since the second term of this is zero by the isotropy of φ, (15) lies in $\theta_1(\varphi)$ and so in $\theta_{k-1}(\varphi)$. It follows that the right-hand side of (14) lies in $\theta_{k-1}(\varphi) + \dfrac{\nabla^k v}{\partial z^k}$ as required, completing the inductive step.

Using (13) and the isotropy of φ we have, for any $r, s \geq 1$,

$$
\begin{aligned}
\frac{\partial}{\partial t}\Big|_0 \eta^{\mathbf{R}}_{r,s}(\varphi_t) &= \left\langle \frac{\nabla}{\partial t}\Big|_0 \frac{\nabla^{r-1}}{\partial z^{r-1}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z}, \frac{\nabla^{s-1}}{\partial z^{s-1}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} + \left\langle \frac{\nabla^{r-1}}{\partial z^{r-1}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z}, \frac{\nabla}{\partial t}\Big|_0 \frac{\nabla^{s-1}}{\partial z^{s-1}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} \right\rangle^{\mathbf{C}} \\
&= \left\langle \frac{\nabla^r v}{\partial z^r}, \frac{\nabla^{s-1}}{\partial z^{s-1}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z} \right\rangle^{\mathbf{C}} + \left\langle \frac{\nabla^{r-1}}{\partial z^{r-1}} \frac{\partial^{\mathbf{c}}\varphi}{\partial z}, \frac{\nabla^s v}{\partial z^s} \right\rangle^{\mathbf{C}} \\
&= j^{\mathbf{R}}_{r,s}(v)
\end{aligned}
$$

as required. $\qquad\square$

Say that a vector field v along a smooth map $\varphi : M^2 \to N$ *preserves real isotropy to first order* if $j^{\mathbf{R}}_{r,s}(v) = 0$ for all $r, s \geq 1$; note that this condition is independent of the choice of local complex coordinate.

Proposition 3.4 *Let $\varphi : S^2 \to N$ be a harmonic map from the 2-sphere to a space form. Then any Jacobi field preserves real isotropy to first order, equivalently, any smooth variation of φ tangent to a Jacobi field is real isotropic to first order.*

Proof Let $\{\varphi_t\}$ be any smooth variation of φ. We shall show that $\eta^{\mathbf{R}}_{r,s}(\varphi_t) = o(t)$ for all integers $r, s \geq 1$. In fact we show by induction that, for all $K \in \{1, 2, \dots\}$,

$$
\text{(i)} \quad \frac{\nabla}{\partial \bar{z}} \frac{\nabla^{K-1}}{\partial z^{K-1}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} \in \theta_{K-1}(\varphi_t) + o(t), \tag{16}
$$

$$
\text{(ii)} \quad \text{for all } r, s \geq 1 \text{ with } r + s = K + 1, \ \eta^{\mathbf{R}}_{r,s}(\varphi_t) = o(t). \tag{17}
$$

This is true for $K = 1$ by (10) and Proposition 3.2.

Suppose that it is true for all $K < k$ for some $k \geq 2$. We shall show that it is true for $K = k$. Indeed,

$$
\frac{\nabla}{\partial \bar{z}} \frac{\nabla^{k-1}}{\partial z^{k-1}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} = R\left(\frac{\partial^{\mathbf{c}}\varphi_t}{\partial z}, \frac{\partial^{\mathbf{c}}\varphi_t}{\partial \bar{z}} \right) \frac{\nabla^{k-2}}{\partial z^{k-2}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} + \frac{\nabla}{\partial z} \frac{\nabla}{\partial \bar{z}} \frac{\nabla^{k-2}}{\partial z^{k-2}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z}. \tag{18}
$$

The curvature term is a multiple of

$$
\left\langle \frac{\partial^{\mathbf{c}}\varphi_t}{\partial \bar{z}}, \frac{\nabla^{k-2}}{\partial z^{k-2}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} \right\rangle^{\mathbf{C}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} - \left\langle \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z}, \frac{\nabla^{k-2}}{\partial z^{k-2}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial z} \right\rangle^{\mathbf{C}} \frac{\partial^{\mathbf{c}}\varphi_t}{\partial \bar{z}}.
$$

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The first term of this lies in $\theta_1(\varphi_t)$ and so in $\theta_{k-1}(\varphi_t)$; by the induction hypothesis, the second term is $o(t)$. Also, by the induction hypothesis, the last term of (18) lies in $\theta_{k-1}(\varphi_t)$, so that (16) holds for $K = k$. Next, for $r, s \geq 1$ with $r + s = k + 1$ we have

$$\frac{\partial}{\partial \bar{z}} \eta_{r,s}^{\mathbf{R}}(\varphi_t) = \Big\langle \frac{\nabla}{\partial \bar{z}} \frac{\nabla^{r-1}}{\partial z^{r-1}} \frac{\partial^{\mathbf{c}} \varphi_t}{\partial z}, \frac{\nabla^{s-1}}{\partial z^{s-1}} \frac{\partial^{\mathbf{c}} \varphi_t}{\partial z} \Big\rangle^{\mathbf{C}} + \Big\langle \frac{\nabla^{r-1}}{\partial z^{r-1}} \frac{\partial^{\mathbf{c}} \varphi_t}{\partial z}, \frac{\nabla}{\partial \bar{z}} \frac{\nabla^{s-1}}{\partial z^{s-1}} \frac{\partial^{\mathbf{c}} \varphi_t}{\partial z} \Big\rangle^{\mathbf{C}}.$$

By the induction hypothesis, this is $o(1)$, hence $\dfrac{\partial}{\partial t}\Big|_0 \eta_{r,s}^{\mathbf{R}}(\varphi_t)\mathrm{d}z^{k+1}$ defines a holomorphic $(k+1)$-differential on S^2. As before, this must vanish, so that $\eta_{r,s}^{\mathbf{R}}(\varphi_t) = o(t)$ completing the induction step. $\qquad\square$

4 Preservation of holomorphicity and complex isotropy to first order

We give two properties of a harmonic map into a Kähler manifold which are preserved to first order under smooth variations tangent to a Jacobi field; the first is holomorphicity which involves only first order partial derivatives, the second is complex isotropy which involves higher order partial derivatives.

4.1 Preservation of holomorphicity

Let M and N be compact Kähler manifolds. Then we can decompose the complexified tangent spaces $T^{\mathbf{c}}M = TM \otimes_{\mathbb{R}} \mathbb{C}$ and $T^{\mathbf{c}}N = TN \otimes_{\mathbb{R}} \mathbb{C}$ into $(1,0)$ and $(0,1)$ parts, viz.,

$$T^{\mathbf{c}}M = T'M \oplus T''M \qquad T^{\mathbf{c}}N = T'N \oplus T''N. \tag{19}$$

We shall let $\langle \, , \, \rangle^{\mathrm{Herm}}$ denote the *Hermitian extension* of the inner product on TN to $T^{\mathbf{c}}N$ given by

$$\langle v, w \rangle^{\mathrm{Herm}} = \langle v, \overline{w} \rangle^{\mathbf{C}};$$

this restricts to a positive definite Hermitian inner product on $T'N$ and on its pull-back $\varphi^{-1}T'N$. Let $\varphi : M \to N$ be a smooth map. As before extend its differential to a complex-linear map $\mathrm{d}^{\mathbf{c}}\varphi : T^{\mathbf{c}}M \to T^{\mathbf{c}}N$; then we can consider the two components:

$$\partial\varphi : T'M \to T'N, \qquad \bar{\partial}\varphi : T''M \to T'N.$$

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Recall that φ is called *holomorphic* if $\bar{\partial}\varphi = 0$ (and *antiholomorphic* if $\partial\varphi = 0$).

For simplicity we calculate in a local complex coordinate system (z^j) on M, writing

$$\frac{\partial\varphi}{\partial z^j} = (\partial\varphi)\left(\frac{\partial}{\partial z^j}\right), \quad \frac{\partial\varphi}{\partial z^{\bar{j}}} = (\bar{\partial}\varphi)\left(\frac{\partial}{\partial z^{\bar{j}}}\right);$$

note that

$$\frac{\partial^{\mathbf{c}}\varphi}{\partial z^j} = \frac{\partial\varphi}{\partial z^j} + \overline{\frac{\partial\varphi}{\partial z^{\bar{j}}}}.$$

Say that a smooth variation $\{\varphi_t\}$ is *holomorphic to first order* if $\bar{\partial}\varphi_t = o(t)$, i.e., for any local complex coordinate system (z^j),

$$\frac{\nabla}{\partial t}\bigg|_0 \frac{\partial\varphi_t}{\partial z^{\bar{j}}} = \frac{\partial\varphi}{\partial z^{\bar{j}}} = 0 \quad \forall j.$$

For any $v \in \Gamma(\varphi^{-1}T^{\mathbf{c}}N)$, let v' denote its $(1,0)$ component under the decomposition (19). Note that, if v is real, i.e., lies in $\Gamma(\varphi^{-1}TN)$, v' determines v, indeed $v = v' + \overline{v'}$. Say that v is *holomorphic* if $\nabla_Z v' = 0$ for all $Z \in T''M$, equivalently,

$$\frac{\nabla v'}{\partial z^{\bar{j}}} = 0 \quad \forall j. \tag{20}$$

Holomorphicity to first order is then equivalent to holomorphicity of the variation vector field as follows (cf. Lemma 3.1):

Lemma 4.1 *Let* $\varphi : M \to N$ *be a holomorphic map between Kähler manifolds and let* $v \in \Gamma(\varphi^{-1}TN)$. *Then the following are equivalent:*

  (i) v *is a holomorphic vector field along* φ;
 (ii) *there exists a smooth variation of* φ *tangent to* v *which is holomorphic to first order;*
(iii) *all smooth variations of* φ *tangent to* v *are holomorphic to first order.*

□

Proof Noting that the Kähler condition on N implies that $(\nabla_Z v)' = \nabla_Z(v')$ for all $v \in \Gamma(\varphi^{-1}T^{\mathbf{c}}N)$, $Z \in T^{\mathbf{c}}M$, we have

$$\frac{\nabla}{\partial t}\bigg|_0 \frac{\partial\varphi_t}{\partial z^{\bar{j}}} = \frac{\nabla}{\partial z^{\bar{j}}}\left(\frac{\partial\varphi_t}{\partial t}\bigg|_0\right)' = \frac{\nabla v'}{\partial z^{\bar{j}}} \quad \forall j$$

from which the lemma follows. □

Proposition 4.2 *Let $\varphi : M \to N$ be a holomorphic map between compact Kähler manifolds. Then any Jacobi field along φ is holomorphic; equivalently, any smooth variation of φ tangent to a Jacobi field is holomorphic to first order.*

Proof The harmonicity condition for a smooth map $\varphi : M \to N$ between Kähler manifolds is (with summation over i and j) $g^{i\bar{j}} \dfrac{\nabla}{\partial z^i} \dfrac{\partial \varphi}{\partial z^{\bar{j}}} = 0$. Hence a smooth variation $\{\varphi_t\}$ of a harmonic map $\varphi : M \to N$ is tangent to a Jacobi field v if and only if

$$\frac{\partial}{\partial t}\Big|_0 \, g^{i\bar{j}} \frac{\nabla}{\partial z^i} \frac{\partial \varphi_t}{\partial z^{\bar{j}}} = 0 \,.$$

On using the curvature to commute the derivatives, this gives the Jacobi equation in the form

$$g^{i\bar{j}} \left\{ \frac{\nabla^2 v'}{\partial z^i \partial z^{\bar{j}}} + R\Big(\frac{\partial^c \varphi}{\partial z^i}, v\Big) \frac{\partial \varphi}{\partial z^{\bar{j}}} \right\} = 0 \,.$$

If φ is holomorphic, $\partial \varphi / \partial z^{\bar{j}} = 0$ so that the curvature term vanishes; then integrating by parts gives

$$\int_M g^{i\bar{j}} \Big\langle \frac{\nabla v'}{\partial z^{\bar{j}}}, \frac{\nabla v'}{\partial z^{\bar{i}}} \Big\rangle^{\mathrm{Herm}} \omega_g = 0 \,.$$

Therefore $\dfrac{\nabla v'}{\partial z^{\bar{j}}} = 0 \;\; \forall j$, so that v is holomorphic. $\qquad\qquad\square$

For an alternative proof see [17].

4.2 Preservation of complex isotropy

Let N be a Kähler manifold. For any smooth map $\varphi : M^2 \to N$ from a Riemann surface, integers $r, s \geq 1$ and complex coordinate z, set

$$\eta^{\mathbf{C}}_{r,s}(\varphi) = \Big\langle \frac{\nabla^{r-1}}{\partial z^{r-1}} \frac{\partial \varphi}{\partial z}, \frac{\nabla^{s-1}}{\partial \bar{z}^{s-1}} \frac{\partial \varphi}{\partial \bar{z}} \Big\rangle^{\mathrm{Herm}} \,.$$

Then φ is called *complex isotropic* if $\eta^{\mathbf{C}}_{r,s}(\varphi) = 0$ for all integers $r, s \geq 1$. Note that this condition is independent of the complex coordinate chosen and that it is stronger than real isotropy.

It is well-known that any harmonic map $\varphi : S^2 \to N$ from the 2-sphere to a complex space form (i.e. a Kähler manifold of constant holomorphic

sectional curvature) is complex isotropic, see, for example, [7, 10, 22]. As in the real case, this is proved by showing inductively that, for $k = 2, 3, \ldots$ and $r + s = k$, $\eta_{r,s}^{\mathbf{C}}(\varphi)\mathrm{d}z^k$ defines a holomorphic k-differential on S^2 and so must vanish.

For any smooth map $\varphi : M^2 \to N$ from a Riemann surface to a Kähler manifold, vector field v along φ, integers $r, s \geq 1$ and local complex coordinate z, set

$$j_{r,s}^{\mathbf{C}}(v) = \left\langle \frac{\nabla^r v}{\partial z^r}, \frac{\nabla^{s-1}}{\partial \bar{z}^{s-1}} \frac{\partial \varphi}{\partial \bar{z}} \right\rangle^{\mathrm{Herm}} + \left\langle \frac{\nabla^{r-1}}{\partial z^{r-1}} \frac{\partial \varphi}{\partial z}, \frac{\nabla^s v}{\partial \bar{z}^s} \right\rangle^{\mathrm{Herm}}.$$

Then, analogously to Lemma 3.3 we have

Lemma 4.3 *Let $\varphi : M^2 \to N$ be a complex isotropic map from a Riemann surface into a space form and let v be a Jacobi field along it. Then, for any smooth variation $\{\varphi_t\}$ of φ tangent to v and any complex coordinate z on M^2,*

$$j_{r,s}^{\mathbf{C}}(v) = \left. \frac{\partial}{\partial t} \right|_0 \eta_{r,s}^{\mathbf{C}}(\varphi_t) \qquad (r, s \geq 1).$$

$\square$

Say that v *preserves complex isotropy to first order* if $j_{r,s}^{\mathbf{C}}(v) = 0$ for all $r, s \geq 1$, and say that $\{\varphi_t\}$ *is complex isotropic to first order* if $\eta_{r,s}^{\mathbf{C}}(\varphi_t) = o(t)$, i.e.,

$$\left. \frac{\partial}{\partial t} \right|_0 \eta_{r,s}^{\mathbf{C}}(\varphi_t) = \eta_{r,s}^{\mathbf{C}}(\varphi) = 0 \qquad (r, s \geq 1).$$

Note that these conditions are independent of the choice of local complex coordinate. Then we have the complex analogue of Proposition 3.4:

Proposition 4.4 *Let $\varphi : S^2 \to N$ be a harmonic map from the 2-sphere to a complex space form. Then any Jacobi field preserves complex isotropy to first order, equivalently, any smooth variation of φ tangent to a Jacobi field is complex isotropic to first order.* $\square$

The proofs proceed in the same way as for the real case, except that we must use the more complicated formula for the curvature of a complex space form (see, for example, [14, Chapter IX]). The calculations are omitted; they are simplified by the following easy consequence of that curvature formula:

Lemma 4.5 *For any $X \in T^{\mathbf{C}}N$, $R\left(X, \frac{\partial^{\mathbf{c}} \varphi}{\partial z} \right) \frac{\nabla^{k-1}}{\partial z^{k-1}} \frac{\partial \varphi}{\partial z}$ is a linear combination of $\frac{\partial \varphi}{\partial z}$, $\frac{\nabla^{k-1}}{\partial z^{k-1}} \frac{\partial \varphi}{\partial z}$ and $\eta_{k,1}^{\mathbf{C}}(\varphi)X'$.* $\square$

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